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Determination of the electro-magnetic forces on two moving charges using the Biot-Savart and Lorentz laws.

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Abstract

A detailed calculation of the magnetic and electric forces acting on two equally charged moving particles is presented. The derivation is based in the use of the magnetic field Biot-Savart law and the magnetic force Lorentz law and is amply supported with graphs.

System definition

Fig. 1 depicts a broad system of two particles with equal negative charge, n_1 and n_2, traveling in opposite directions identified with vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively; the centers of the particles are located on a reference axis of length $s_{1,2}$ which is used to define the particles direction angles, namely, θ_1 and θ_2 which are chosen such that the former is positive and the latter is negative. The angles between the speed vectors are identified as β_1 and β_2; these are equal and are given by 180 - θ_1 - θ_2, in fact, they will not be required for the derivation further.

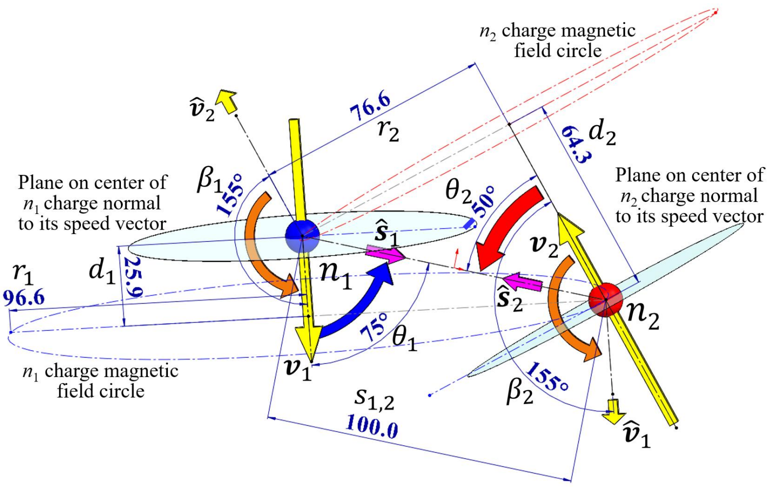

Fig. 1. A comprehensive two moving negative charges system definition.

At each particle center, a plane is outlined perpendicular to the corresponding particle speed vector; for each of them, another parallel plane is positioned at distances d_1 and d_2 where the magnetic field circle of each particle is delineated, see radii r_1 and r_2, in such a way that the magnetic interaction with the other particle center is produced. Note that each particle center point is located in two of the described planes.

Magnetic field vectors on each particle. The Biot-Savart law.

In order to determine the magnetic field vectors acting on the particles, we will make use of Fig 2.

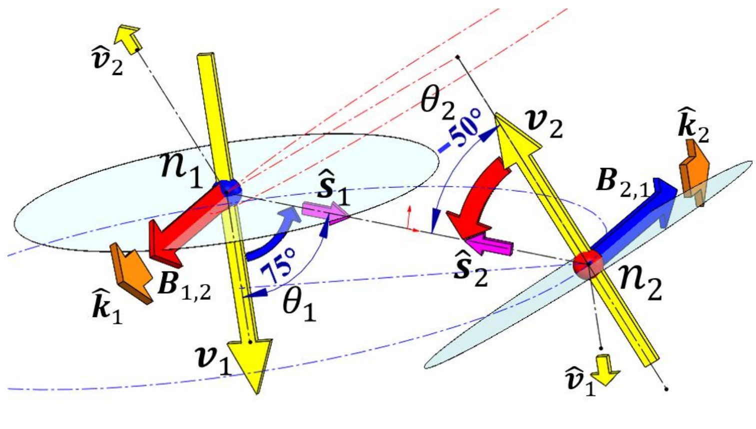

Fig. 2. The particles reciprocally interacting magnetic fields determined by the Biot-Savart law.

The Biot-Savart law states that the magnetic field acting on particle n_1 produced by particle n_2 is given by

$$\boldsymbol{B}_{1,2} = \frac{\mu_0 n_2}{4\pi s_{1,2}^2}\boldsymbol{v}_2 \times \hat{\boldsymbol{s}}_2 \;\; = \frac{\mu_0 n_2}{4\pi s_{1,2}^2}\{|v_2||1|\,sin\,(\theta_2)\,\widehat{\boldsymbol{k}}_1\}(\rightarrow)_{\text{RHR}} \qquad (1)$$

where μ_0 ($N \cdot A^{-2}$) is the vacuum magnetic permeability while $\hat{\boldsymbol{s}}_2$ and $\widehat{\boldsymbol{k}}_1$ are directional unit vectors; the curly brackets enclose the vector cross product definition but the expression is not yet fully evaluated; its direction is indicated as being frontwards ($\rightarrow$),that is, as the right-hand rule points toward, namely, the $\boldsymbol{v}_2$ play the fingers role, $\hat{\boldsymbol{s}}_2$ gives the palm direction and the finger closing sense is represented by the θ_2 arrow. Evaluating the sine term and taking care of the charge sign with its absolute value gives two direction reversals of the resulting vector ($\leftarrow$, $\rightarrow$), so we obtain

$$\boldsymbol{B}_{1,2} = 0.766\frac{|n_2|\mu_0}{4\pi s_{1,2}^2}v_2\widehat{\boldsymbol{k}}_1(\rightarrow) \qquad (2)$$

Keep in mind that this vector location would be distant from $\boldsymbol{v}_2$ and $\hat{\boldsymbol{s}}_2$ meeting point.

Concomitantly, the magnetic field vector on particle n_2 generated by particle n_1 is

$$\boldsymbol{B}_{2,1} = \frac{\mu_0 n_1}{4\pi s_{1,2}^2}\boldsymbol{v}_1 \times \hat{\boldsymbol{s}}_1 = \frac{\mu_0 n_1}{4\pi s_{1,2}^2}\{|v_1|\,sin\,(\theta_1)\,\widehat{\boldsymbol{k}}_2\}(\rightarrow)_{\text{RHR}} \qquad (3)$$

where $\hat{\boldsymbol{s}}_1$ and $\widehat{\boldsymbol{k}}_2$ are also directional unit vectors. As before, the curly bracketed vector cross product is yet to be completed. The magnetic field vector reversal from the original RHR orientation due to the n_1 sign produces

$$\boldsymbol{B}_{2,1} =- 0.966\frac{|n_1|\mu_0}{4\pi s_{1,2}^2}v_1\widehat{\boldsymbol{k}}_2(\leftarrow) \qquad (4)$$

whose direction is opposite to $\boldsymbol{B}_{1,2}$. The (3) and (4) magnitude ratio is

$$|\boldsymbol{B}_{1,2}|/|\boldsymbol{B}_{2,1}| \approx 0.793(^{v_2}/_{v_1}) \qquad (5)$$

Magnetic force vector on each particle. The Lorentz law.

On what follows, we'll make use of Fig. 3. Applying the Lorentz magnetic force law on particle n_1 gives

$$\boldsymbol{F}_1 = n_1\boldsymbol{v}_1 \times \boldsymbol{B}_{1,2} = n_1\{|v_1||B_{1,2}|sin(\gamma_1)\hat{\boldsymbol{j}}_1\}(\leftarrow)_{\text{RHR}} \qquad (6)$$

Then, using $\gamma_1 = 90°$ and expression (2), this turns into

$$\boldsymbol{F}_1 =- |n_1||v_1||B_{1,2}|(1)\hat{\boldsymbol{j}}_1(\rightarrow) =- 0.766\frac{\mu_0|n_1||n_2|v^2}{4\pi s_{1,2}^2}\hat{\boldsymbol{j}}_1(\rightarrow) \qquad (7)$$

where the charge sign vector sense reversal was included and $v = v_1 = v_2$ was introduced. $\hat{\boldsymbol{j}}_1$ is a directional unit vector whose path will be described shortly.

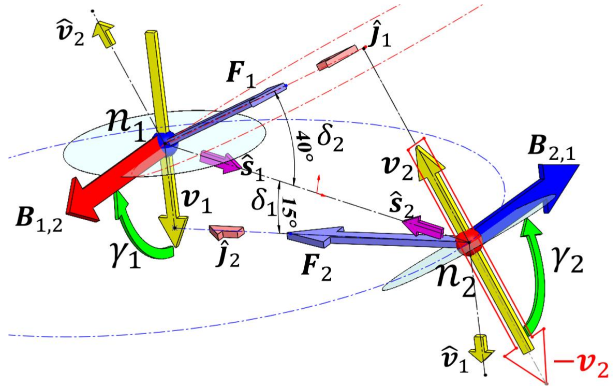

Fig. 3. The magnetic force acting on each charge as established by to the Lorentz law. $\delta_1 = 90 - \theta_1$ and $\delta_2 = 90 - |\theta_2|$.

On particle n_2, the force is

$$\boldsymbol{F}_2 = n_2(-\boldsymbol{v}_2) \times \boldsymbol{B}_{2,1} = n_2\{|v_2||B_{2,1}|sin(\gamma_2)\hat{\boldsymbol{j}}_2\}(\rightarrow)_{\text{RHR}} \quad (8)$$

where the $\boldsymbol{v}_2$ negative sign was taken care of by switching its direction; $\gamma_2 = 90°$ is kept. With (4) and the charge sign reversal considered, (8) becomes

$$\boldsymbol{F}_2 =-0.966\frac{\mu_0|n_1||n_2|v^2}{4\pi s_{1,2}^2}\hat{\boldsymbol{j}}_2(\leftarrow) \quad (9)$$

where $\hat{\boldsymbol{j}}_2$ is another directional unit vector.

Making $n_1 = n_2 = n$, multiplying and dividing (7) by the square of the light speed, c^2, allows us to derive

$$\boldsymbol{F}_1 =-0.766\frac{n^2\mu_0 c^2}{4\pi s_{1,2}^2}\left(\frac{v}{c}\right)^2\hat{\boldsymbol{j}}_1 =-0.766\frac{n^2}{4\pi\varepsilon_0 s_{1,2}^2}\hat{\boldsymbol{j}}_1 \quad (10)$$

or

$$\boldsymbol{F}_1 =-0.766F_C\hat{\boldsymbol{j}}_1 = \sin(\theta_2)F_C\hat{\boldsymbol{j}}_1 \quad (11)$$

where we made $v = c$, used $\mu_0 c^2 = 1/\varepsilon_0$ and defined the electric or Coulomb force among the two particles as

$$F_C =+\frac{n^2}{4\pi\varepsilon_0 s_{1,2}^2}\ (\text{N}) \quad (12)$$

here, ε_0 ($C^2 \cdot kg^{-1} \cdot m^{-3} \cdot s^2$) is the vacuum permittivity.

Similarly, (9) becomes

$$\boldsymbol{F}_2 =-\sin(\theta_1)F_C\hat{\boldsymbol{j}}_2 \quad (13)$$

Note that both particle vector forces are radially directed into the center of the magnetic field circle produced by the opposite particle; the plane inclinations with respect to the reference axis are δ_2 for $\boldsymbol{F}_1$ and δ_1 for $\boldsymbol{F}_2$, These angles are defined in Fig. 3 as $\delta_1 = 90 - \theta_1$ and $\delta_2 = 90 - |\theta_2|$.

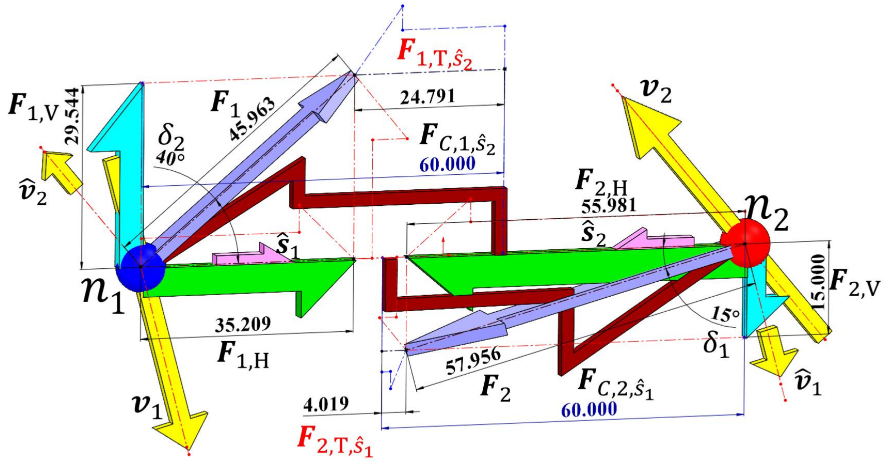

Fig. 4. The magnetic force orthogonal components derivation and the Coulomb force vectors incorporation.

The Coulomb force and the magnetic force components.

For the sake of plotting clearness, let's assign to (11) and (13) a Coulomb force term of 60 units so that their respective magnitude become of 45.963 and 55.981 units, as can be seen in Fig. 4 on solid arrows sideways with the particle Coulomb forces $\boldsymbol{F}_{C,1,\hat{s}_2}$ and $\boldsymbol{F}_{C,2,\hat{s}_1}$. The two magnetic orthogonal components for each force are plotted as solid arrows identified as $\boldsymbol{F}_{1,V}$ and $\boldsymbol{F}_{1,H}$ acting on particle n_1 and $\boldsymbol{F}_{2,V}$ and $\boldsymbol{F}_{2,H}$ on particle n_2. There hollow arrow versions form a right triangle with each magnetic force so that their components are calculated as follows

$$\boldsymbol{F}_{1,\mathrm{H}} = -\cos(\delta_2)\, F_1 \hat{\boldsymbol{s}}_1 = -\sin(\theta_2)\, F_1 \hat{\boldsymbol{s}}_1 =- sin^2(\theta_2)\, F_C \hat{\boldsymbol{s}}_1 =- 35.209\hat{\boldsymbol{s}}_1 \qquad (14)$$

and

$$\boldsymbol{F}_{1,\mathrm{V}} =- \sin(\delta_2) F_1 \hat{\boldsymbol{z}}_1 = -\cos(\theta_2)\sin(\theta_2) F_C \hat{\boldsymbol{z}}_1 = -0.5\sin(2\theta_2) F_C \hat{\boldsymbol{z}}_1 =- 29.544\hat{\boldsymbol{z}}_1 \qquad (15)$$

for n_1, where $\hat{\boldsymbol{z}}_1$ is an up vertical unit vector perpendicular to $\hat{\boldsymbol{s}}_1$ and

$$\boldsymbol{F}_{2,\mathrm{H}} =- \cos(\delta_1)\, F_2 \hat{\boldsymbol{s}}_2 =- \sin^2(\theta_1)\, F_C \hat{\boldsymbol{s}}_2 =- 55.981\hat{\boldsymbol{s}}_2 \qquad (16)$$

and

$$\boldsymbol{F}_{2,\mathrm{V}} = -\sin(\delta_1)\, F_2 \hat{\boldsymbol{z}}_2 = -\cos(\theta_1)\sin(\theta_1)\, F_C \hat{\boldsymbol{z}}_2 = -0.5\sin(2\theta_1)\, F_C \hat{\boldsymbol{z}}_2 = -15.0\hat{\boldsymbol{z}}_2 \quad (17)$$

for n_2; $\hat{\boldsymbol{z}}_2$ is a down vertical unit vector perpendicular to $\hat{\boldsymbol{s}}_2$.

The total forces.

Now, on (14) and (16) we ought to take into account the involved Coulomb force on each particle so the total horizontal force on n_1 is

$$\boldsymbol{F}_{1,\mathrm{T},\hat{s}_2} = \boldsymbol{F}_{C,1,\hat{s}_2} + \boldsymbol{F}_{1,\mathrm{H}} = \boldsymbol{F}_C \cos^2(\theta_2) = +\, 24.791\hat{\boldsymbol{s}}_2 \quad (18)$$

and on particle n_2 is

$$\boldsymbol{F}_{2,\mathrm{T},\hat{s}_1} = \boldsymbol{F}_{C,2,\hat{s}_1} + \boldsymbol{F}_{2,\mathrm{H}} = \boldsymbol{F}_C \cos^2(\theta_1) = +\, 4.019\hat{\boldsymbol{s}}_1 \quad (19)$$

as is shown in Figs. 4 and 5.

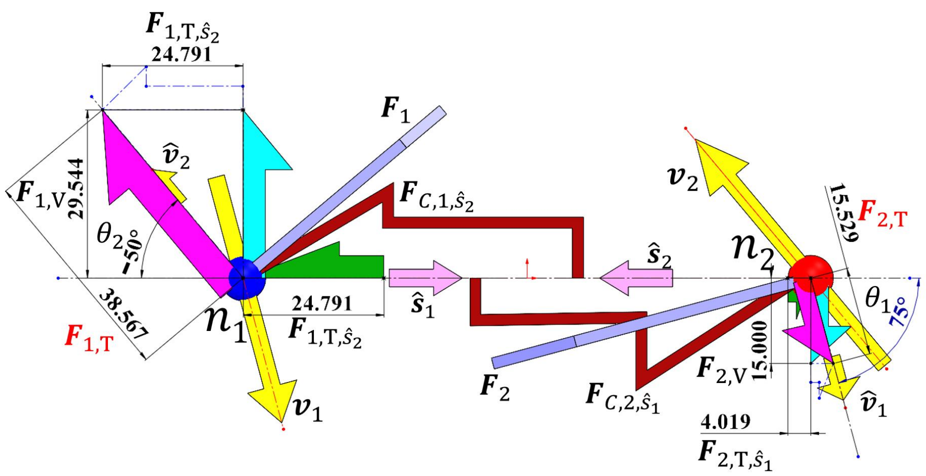

Fig. 5. The total electro-magnetic force $F_{1,T}$ and $F_{2,T}$.

Finally, adding the horizontal and vertical components acting on particle n_1, its total force is

$$\boldsymbol{F}_{1,\mathrm{T}} = \boldsymbol{F}_{1,\mathrm{T},\hat{s}_2} + \boldsymbol{F}_{1,\mathrm{V}} = -\sqrt{F_{1,\mathrm{T},\hat{s}_2}^2 + F_{1,\mathrm{V}}^2}\; \hat{\boldsymbol{v}}_2 = -\, 38.567\hat{\boldsymbol{v}}_2 \quad (20)$$

and on particle n_2, the total force is

$$\boldsymbol{F}_{2,\mathrm{T}} = \boldsymbol{F}_{2,\mathrm{T},\hat{s}_1} + \boldsymbol{F}_{2,\mathrm{V}} = -\sqrt{F_{2,\mathrm{T},\hat{s}_1}^2 + F_{2,\mathrm{V}}^2}\; \hat{\boldsymbol{v}}_1 = -\, 15.529\hat{\boldsymbol{v}}_1 \quad (21)$$

as is shown in Fig. 5.

Expressions (20) and (21) can also be written as

$$\boldsymbol{F}_{1,\mathrm{T}} = -\cos(\theta_2) F_C \, \widehat{\boldsymbol{v}}_2 = -\, 38.567 \widehat{\boldsymbol{v}}_2 \tag{22}$$

and

$$\boldsymbol{F}_{2,\mathrm{T}} = -\cos(\theta_1) F_C \, \widehat{\boldsymbol{v}}_1 = -\, 15.529 \widehat{\boldsymbol{v}}_1 \tag{23}$$

Also, the direction of these vectors is verified as follows

$$\delta_2 = atan(F_{1,\mathrm{T},\hat{s}_2}/F_{1,\mathrm{V}})(180/\pi) = 40 \tag{24}$$

and

$$\delta_1 = atan(F_{2,\mathrm{T},\hat{s}_1}/F_{2,\mathrm{V}})(180/\pi) = 15 \tag{25}$$

Table 1 provides a summary of the used input data and the derived ones along with the manuscript location where they were introduced or calculated.

Table 1. Input and output data with their corresponding manuscript definition.

θ_1	θ_2	δ_2	δ_1	F_C	F_1	F_2	$F_{1,H}$	$F_{2,H}$	$F_{1,V}$	$F_{2,V}$	$F_{1,T,s2}$	$F_{2,T,s1}$	$F_{1,T}$	$F_{2,T}$	δ_2	δ_1
75	-50	40	15	60	-45.96	-57.96	-35.21	-55.98	-29.54	-15.00	24.79	4.02	-38.57	-15.53	40	15
Fig. 1	Fig. 1	Fig. 3	Fig. 3	Fig. 4	(11)	(13)	(14)	(16)	(15)	(17)	(18)	(19)	(22)	(23)	(24)	(25)

Adding (14) and (15) gives

$$\boldsymbol{F}_1 = -F_C(\cos(\theta_2)\sin(\theta_2)\hat{\boldsymbol{z}}_1 + sin^2(\theta_2)\,\hat{\boldsymbol{s}}_1) = -F_C(\cos(\theta_2)\widehat{\boldsymbol{v}}_2 + sin^2(\theta_2)\,\hat{\boldsymbol{s}}_1) \tag{26}$$

and adding now (16) and (17)

$$\boldsymbol{F}_2 = -F_C(\cos(\theta_1)\sin(\theta_1)\hat{\boldsymbol{z}}_2 + sin^2(\theta_1)\,\hat{\boldsymbol{s}}_2) = -F_C(\cos(\theta_1)\widehat{\boldsymbol{v}}_1 + sin^2(\theta_1)\,\hat{\boldsymbol{s}}_2) \tag{27}$$

These expressions are the expanded versions of (11) and (13).

Conclusions.

A set of equations was derived to calculate the magnetic forces and their components acting on two negatively charged, moving particles.

As can be seen in Fig. 5, the interaction of the moving charges produces a sort of torque effect complemented with a slowdown result on each of them. The electrical and magnetic total force on each particle is of attraction and is directed along the velocity vector of the other one.

The described behavior suggests that the magnetic force differs radically from the electrostatic force. Firstly, the two magnetic forces are negative, that is, the particles attract each other while the electrostatic force on each particle is positive and hence, they repel each other. Secondly, the net magnitude of the magnetic forces can be different as seen above, while for the electrostatic force, both particle forces have to be equal in magnitude. Lastly, electrostatic forces are always collinear and act along the shortest path joining the charges while the magnetic force on each particle acts radially on the magnetic field circle. They can be equal and act along the reference joining axis only when the particles paths are parallel or antiparallel, that is, when $\theta_1 = \theta_2 = \pm 90°$; even for these cases, they keep the negative or attracting nature for the two negative charges case; essentially, it happens the same for two positive charges case. For the extreme case of particles moving in a collinear path along the reference joining axis, it happens that $\theta_1 = \theta_2 = 0°$ or $\theta_1 = \theta_2 = 180°$, so that the sine term vanishes and so do the magnetic field and, consequently, the force on each particle.

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Conflict of Interest Statement

The author states that there are no conflicts of interest with respect to the funding, research, authorship and publication of this article.

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