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BOUNDARY VALUE PROBLEMS ON THERMAL STAR GRAPH AND THEIR SOLUTIONS

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Abstract.

Boundary value problems of thermal conductivity on a star graph are considered, inspired by engineering applications, e.g., heat conduction phenomena in mesh-like structures. Based on the generalized function method, a unified technique for solving boundary value problems of heat conduction has been developed. Generalized solutions to transient and stationary boundary value problems are constructed for several types of boundary conditions at the ends, with the generalized Kirchhoff conditions at the node. Using the symmetry properties of the Fourier transform of the fundamental solution, regular integral representations of solutions to boundary value problems are obtained.

The derived results allow simulation of heat sources of various types, including those involving singular generalized functions. The employed method of generalized functions enables tackling a wide class of boundary value problems, including those with local and connected boundary conditions at the ends of the graph, and various transmission conditions at the node.

Key words: uncoupled thermoelastodynamics, thermal graph, elastic graphs, generalized functions method, Fourier transformation, resolving system for boundary functions, stationary oscillation.

Introduction

With the development of mechanical engineering, complex multi-link rod structures operating under various thermal conditions began to be actively used. They are widely used in structural mechanics, mechanical engineering, robotics and many other fields. An urgent scientific and technical task is to study the thermally stressed state of network systems for various purposes under dynamic and thermal influences, taking into account their thermoelastic properties under dynamic and thermal influences, including impact types. This is necessary to analyze the structural strength and reliability of such objects, determine safe operating modes and prevent disasters. Mathematical modeling of the thermodynamics of rod structures and the creation of information technologies based on it is one of the more effective and inexpensive methods for researching and designing such systems.

Graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation [1,2]. The properties of graphs are also actively used to solve boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks [3-10].

Here boundary value problems of uncoupled thermoelasticity are considered on a star thermoelastic graph (Fig.1), which can be used to study various mesh structures under conditions of thermal heating. Based on the generalized function method, a unified technique has been developed for solving boundary value problems of uncoupled thermoelasticity, typical for engineering applications. Generalized solutions to nonstationary and stationary boundary value problems of uncoupled thermoelasticity on a stellar graph are constructed under various boundary conditions at the ends of the graph and generalized Kirchhoff conditions at its common node. Regular integral representations of solutions to boundary value problems are obtained in analytical form. The solutions obtained make it possible to simulate force and heat sources of various types, including using singular generalized functions.

Presented here the method of generalized functions (GFM) makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and different transmission conditions at its node.

1. Statement of a boundary value problems on a thermal star graph

We consider an thermal star graph which contains N edges (A_0 , A_j) of the length L_j $(j = 1, 2, ..., N)$ with a common node A_0 (Fig. 1). On each edge $S_j = \{x \in R^1 : 0 \le x \le L_j\}$ there is own coordinate system (x_j, t) with the origin at point A_0 : $x = 0$. A temperature $\theta_j(x,t)$ satisfy the heat conduction equation at S_j :

$$\frac{\partial \theta_j}{\partial t} - \kappa_j \frac{\partial^2 \theta_j}{\partial x^2} = F_j(x,t). \qquad (1)$$

Here κ_j is the thermal diffusivity coefficient on the j-th segment, $F_j(x,t)$ describes the action of heat source, $\theta_1^j(t), \theta_2^j(t)$ are the temperature in the ends of the *j*-th edge.

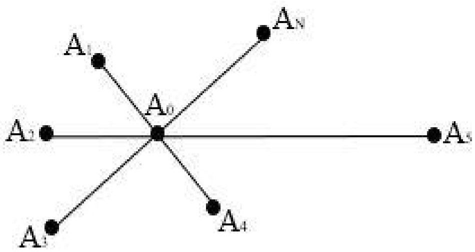

Figure 1. Star graph

The initial conditions at $t = 0$ for the temperature of a graph are known:

(*Cauchy conditions*)

$$\theta_j(x,0) = \theta_0^{\,j}(x), \quad 0 \le x \le L_j, \quad t = 0,$$
$$\theta_j(0) = \theta_0, \ \forall j, \quad \theta_0^{\,j}(x) \in C^2(R^1) \qquad (2)$$

Here we consider the following boundary value problem (BVP).

Dirichlet problem. Temperature values are known at the ends of the graph edges: for all $j=1,...,N$

$$\theta_j(L_j,t)=\theta_2^j(t), \quad t\ge 0, \quad \theta_2^j(t)\in C(R_+^1), \tag{3}$$

$R_+^1=\{t\in[0,\infty)\}$. The following continuity conditions and generalized Kirchhoff conditions are specified in the common node A_0 of the graph:

$$\theta_1^1(t)=\theta_1^2(t)=...=\theta_1^N(t), \quad t\ge 0, \quad \theta_1^j(0)=\theta_0, \tag{4}$$

$$\sum_{j=1}^{N}\kappa_j q_1^j(t)=G(t), \quad G(t)\in L_1(R_+^1), \tag{5}$$

$$\sum_{j=1}^{N}\kappa_j \left.\frac{\partial\theta_0^j}{\partial x}\right|_{x_j=0}=G(0).$$

Here $\theta_1^j(t)=\theta_j(0,t)$, $q_1^j(t)=\left.\frac{\partial\theta_j}{\partial x}\right|_{x=0}$, $q_2^j(t)=\left.\frac{\partial\theta_j}{\partial x}\right|_{x=L_j}$, θ_0 is initial temperature in the common node A_0.

We must to find the solution of this problem on this star graph.

For solving this problem we use the solutions of BVPs for heat equation on segment.

2. Statement of BVP on a graph edge

At first we construct the solution of BVP on one graph segment to get the equations of connection between the boundary temperatures and heat flows on the edge of graph.

Let consider $\theta(x,t)$ on $[0,L]$, which is the solution of heat equation [11]:

$$\frac{\partial\theta}{\partial t}-\kappa\frac{\partial^2\theta}{\partial x^2}=F(x,t). \tag{6}$$

Initial conditions: the temperature is known at $t=0$:

$$\theta(x,0)=\theta_0(x), \quad \theta_0(x)\in C\{0\le x\le L\} \tag{7}$$

Here we construct solutions to BVPs with local and associated boundary conditions.

Local boundary conditions:

$$\begin{cases}\left(\alpha_1\theta_1+\beta_1\Pi_1(t)\right)\big|_{x=0}=G_1(t),\\ \left(\alpha_2\theta_2+\beta_2\Pi_2(t)\right)\big|_{x=L}=G_2(t).\end{cases} \tag{8}$$

where α_j,β_j arbitrary constants, $\theta_j(t),\Pi_j(t)=-k\left.\dfrac{\partial\theta}{\partial x}\right|_{x=x_j}$ $(j=1,2)$ are the temperature and heat flow at ends of the segment in points: $x=x_1=0, x=x_2=L.$ $G_j(t)$ are known function which are integrable functions on R_+^1: $G_j(t)\in L_1(R_+^1)$.

We consider general case of boundary conditions on the ends of graph edges:

$$\alpha_{1j}\theta_1(t)+\beta_{1j}\Pi_1(t)+\alpha_{2j}\theta_1(t)+\beta_{2j}\Pi_2(t)=D_j(t), \quad j=1,2. \tag{9}$$

where α_{ij}, β_{ij} are arbitrary constants. Relations (9) contain all classical formulations of heat BVPs if to take some of constants to be equal zero.

There are initial and boundary conditions which are known:

$$\theta_1(t)=\theta(0,t), \quad \theta_2(t)=\theta(L,t), \quad \theta_j(t)\in C(R_1^+).$$

It is assumed that all functions defining boundary conditions also belong to Lebesgue space L_1.

Let us find solutions to BVPs using Generalized Function Method [12].

2.1. Generalized solution of BVPs on an graph segment. Generalized function method.

To determine the solution we consider BVP in the space of generalized functions of slow growth $S'(R^2)=\{\hat f(x,t), \quad (x,t)\in R^2\}$ [14]. To do this, we introduce a regular generalized function (we mark it with a cap):

$$\hat\theta(x,t)=\begin{cases}\theta(x,t), (x,t)\in D^-\\ 0, \quad x\notin D^-\end{cases},$$

where $\theta(x,t)$ is the solution of BVP, $D^-=[0,L]\times[0,\infty)$. It can be represented in the form

$$\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t).$$

Here $H(x)$ is Heaviside step function.

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we find generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial \hat{\theta}}{\partial x} = \frac{\partial \theta}{\partial x} H(L-x)H(x)H(t) - \theta_2(t)\delta(L-x)H(x)H(t) + \theta_1(t)\delta(x)H(L-x)H(t),$$

$$\frac{\partial^2 \hat{\theta}}{\partial x^2} = \frac{\partial^2 \theta}{\partial x^2} H(L-x)H(x)H(t) - q_2(t)\delta(L-x)H(x)H(t) + q_1(t)H(L-x)\delta(x)H(t) +$$

$$+\theta_2(t)\delta'(L-x)H(x)H(t) + \theta_1(t)H(L-x)H(t)\delta'(x),$$

$$\frac{\partial \hat{\theta}}{\partial t} = \frac{\partial \theta}{\partial t} H(L-x)H(x)H(t) + \theta_0(x)H(L-x)\delta(t),$$

where $\delta(x)$ is singular δ- function, $q_j(t) = \left.\frac{\partial \theta}{\partial x}\right|_{x=x_j}, j=1,2.$

The equation (6) in $S'(R^2)$ has the next form for $\hat{\theta}(x,t)$:

$$\frac{\partial \hat{\theta}}{\partial t} - \kappa \frac{\partial^2 \hat{\theta}}{\partial x^2} = \hat{F}_2(x,t) + \kappa q_2(t)\delta(L-x)H(x)H(t) - \kappa q_1(t)H(L-x)\delta(x)H(t) -$$

$$-\kappa\theta_2(t)\delta'(L-x)H(x)H(t) - \kappa\theta_1(t)\delta'(x)H(L-x)H(t) + \theta_0(x)H(L-x)H(x)\delta(t). \quad (10)$$

Note that the right side of this equation includes all initial and boundary temperature $\theta_j(t)$ and heat flows $\Pi_j(t) = \kappa q_j(t)$ (j=1, 2).

According to the theory of generalized functions [19, 20], the solution of Eq. (10) can be represented as a convolution of fundamental solution of heat equation (6) with the right-hand side of this equation:

$$\hat{\theta}(x,t) = \hat{F}_2(x,t) * U(x,t) + \kappa q_2(t)H(x)H(t) \underset{t}{*} U(L-x,t) -$$

$$-\kappa q_1(t)H(L-x)H(t) \underset{t}{*} U(x,t) - \kappa\theta_2(t)H(t)H(x) \underset{t}{*} U_{,x}(L-x,t) - \quad (11)$$

$$-\kappa\theta_1(t)H(L-x)H(t) \underset{t}{*} U_{,x}(x,t) + \theta_0(x)H(L-x)H(x) \underset{x}{*} U(x,t).$$

Here $\hat{F}(x,t)=F(x,t)H(x)H(L-x)H(t)$, $U\ (x,t)$ is the fundamental solution of the heat equation (1) by $F(x,t)=\delta(x,t)=\delta(x)\delta(t)$, which decays at ∞ and $U(x,t)=\dfrac{\partial U}{\partial x}$.

It has the form [14]:

$$U(x,t)=\frac{1}{\sqrt{2\pi\kappa t}}\exp(-x^2/4\kappa t)H(t). \tag{12}$$

If $\hat{F}(x,t)$ is a regular function, then relation (11) can be represented in the integral form:

$$\theta(x,t)H(L-x)H(x)H(t)= \tag{13}$$

$$=H(t)\int_0^t d\tau\int_{-\infty}^{+\infty}U\left(x-y,t-\tau\right)F_2(y,\tau)dy+\kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau-$$

$$-\kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau-\kappa H(x)H(t)\int_0^t \theta_2(t-\tau)U,_x(L-x,\tau)d\tau-$$

$$-\kappa H(L-x)H(t)\int_0^t U,_x(x,t-\tau)\theta_1(\tau)d\tau+\int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy\ .$$

Formula (13) determines the temperature inside a segment by known temperature and heat flows at its ends and is very useful for engineering applications. However, for correctly posed boundary value problems, out of 4 boundary functions on the right side of formula (13), only 2 are known. To determine two unknown boundary functions, resolving boundary equations should be constructed using boundary conditions at the ends of the segment.

2.2. Solving BVP in the space of Fourier transformation in time. To construct the resolving system of equations on segment, we use Fourier transformation in time:

$$\begin{gathered}\overline{\theta}(x,\omega)=\mathrm{F}\left[\hat{\theta}(x,t)\right]=H(x)H(L-x)\int_0^\infty \theta(x,t)e^{i\omega t}dt,\\ \hat{\theta}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\theta}(x,\omega)e^{-i\omega t}d\omega.\end{gathered} \tag{14}$$

To define Fourier transform of generalized solution (10) we use the property of Fourier transform of convolution [13]:

$$\hat{\theta}(x,\omega) = \overline{F}_2(x,\omega)\underset{x}{*}\overline{U}(x,\omega) + \theta_0(x)H(L-x)H(x)\underset{x}{*}\overline{U}(x,\omega) +$$

$$+\kappa\overline{q}_2(\omega)H(x)\overline{U}(L-x,\omega) - \kappa\overline{q}_1(\omega)H(L-x)\overline{U}(x,\omega) - \tag{15}$$

$$-\kappa\overline{\theta}_2(\omega)H(x)\overline{U}_{,x}(L-x,\omega) - \kappa\overline{\theta}_1(\omega)H(L-x)\overline{U}_{,x}(x,\omega).$$

Here a variable under a sign of convolution shows the convolution only over this variable $\left(\underset{x}{*}\right)$, $\overline{U}_{,x}(x,\omega) = \dfrac{\partial\overline{U}(x,\omega)}{\partial x}$. The integral representation of Eq (15) has the form:

$$\overline{\theta}(x,\omega)H(L-x)H(x)H(\omega) =$$

$$= H(x)\int_0^L \overline{U}\left(x-y,\omega\right)F_2(y,\omega)dy + \kappa H(x)\int_0^L \overline{U}(x-y,\omega)\theta_0(y)dy + \tag{16}$$

$$+\kappa\overline{q}_2(\omega)H(x)\overline{U}(L-x,\omega) - \kappa\overline{q}_1(\omega)H(L-x)\overline{U}(x,\omega) -$$

$$-\kappa\overline{\theta}_2(\omega)H(x)\overline{U}_{,x}(L-x,\omega) - \kappa\overline{\theta}_1(\omega)H(L-x)\overline{U}_{,x}(x,\omega).$$

Fourier transform of Green's function of heat equation is equal to 22

$$\overline{U}(x,\omega) = -\frac{\sin\left(k\left|x\right|\right)}{2k\kappa}, \tag{17}$$

where $k = \sqrt{i\omega\kappa^{-1}} = e^{i\pi/4}\sqrt{\omega\kappa^{-1}} = (1+i)\sqrt{\dfrac{\omega}{2\kappa}}$. It satisfies the equation:

$$\frac{d^2\overline{U}}{dx^2} + i\omega\kappa^{-1}\overline{U} = \delta(x).$$

Its derivative has the gap in point x=0 and equal to

$$\overline{U}_{,x}(x,\omega) = -\frac{\operatorname{sgn} x}{2\kappa}\cos(\kappa\left|x\right|), \quad \operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

There are next symmetry conditions:

$$\bar{U}(x,\omega)=\bar{U}(-x,\omega), \quad \bar{U}_{,x}(\pm 0,\omega)=\mp\frac{1}{2\kappa}. \tag{18}$$

We use these properties for solving BVP.

2.3. Resolving equations of boundary value problems. To find unknown boundary functions, we pass in relation (16) to the limit at $x \to 0+\varepsilon, \quad \varepsilon > 0$:

$$\bar{\theta}_1(\omega)=\lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega)=\bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0}+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0}+$$
$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-0-\varepsilon,\omega)-\kappa\bar{q}_1(\omega)H(L-x)\bar{U}(0+\varepsilon,\omega)-$$
$$-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_x(L-0-\varepsilon,\omega)-\kappa\bar{\theta}_1(\omega)H(L-x)\bar{U}_{,x}(0+\varepsilon,\omega).$$

Next, we move the last term to the left side and taking into account the right limit of $\bar{U}_{,x}(x,\omega)$ at zero (18) . We obtain the next equation on left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega)=\bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0}+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=0}+$$
$$+\kappa\bar{q}_2(\omega)H(x)\bar{U}(L,\omega)-\kappa\bar{q}_1(\omega)\bar{U}(0,\omega)-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L,\omega)$$

Similarly, we consider the limit at $x \to L-\varepsilon, \varepsilon > 0$.

$$\bar{\theta}_2(\omega)=\lim_{\varepsilon\to 0}\bar{\theta}(L-\varepsilon,\omega)=\bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L}+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L}-$$
$$-\kappa\bar{q}_1(\omega)\bar{U}(L-\varepsilon,\omega)-\kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L-\varepsilon,\omega)-\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(\varepsilon,\omega)$$

We move the last term to the left side, and obtain the second boundary equation:

$$\frac{1}{2}\bar{\theta}_2(\omega)=\bar{F}(x,\omega)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L}+\theta_0(x)H(L-x)H(x)\underset{x}{*}\bar{U}(x,\omega)\Big|_{x=L}-$$
$$-\kappa\bar{q}_1(\omega)\bar{U}(L,\omega)-\kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(L,\omega)$$

Let us formulate the obtained results in the form of this theorem.

Theorem 1. *The Fourier time transformants of boundary functions of boundary value problems* (6)-(9) *satisfy the system of linear algebraic equations of the form:*

$$\begin{bmatrix} 0{,}5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \\ + \begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0{,}5 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix}, \tag{19}$$

where

$$\bar{Q}_1(0,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\Big|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\Big|_{x=0},$$

$$\bar{Q}_2(L,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)\Big|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)\Big|_{x=L}.$$

The resulting system (19) makes possibility to solve BVP for any given two boundary functions of temperature and heat flow at the ends of a segment of four boundary functions.

To solve all temperature BVPs, it is convenient to consider the extended system of equations in the form of matrix equation:

$$\mathrm{A}(\omega) \cdot \mathrm{B}(\omega) = \mathrm{C}(\omega), \tag{20}$$

where

$$\mathrm{A}(\omega) = \begin{pmatrix} 0{,}5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0{,}5 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$B(\omega) = \left(\bar{\theta}_1(\omega), \bar{q}_1(\omega), \bar{\theta}_2(\omega), \bar{q}_2(\omega)\right),$$

$$\mathrm{C}(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega), \bar{b}_3(\omega), \bar{b}_4(\omega)).$$

The last two equations in the system (20) are determined by boundary conditions at the ends of the segment, which are known for BVP ((8) or (9)):

$$\begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix}. \tag{21}$$

We have four equations (20) for four boundary functions. By given coefficients a_{ij} and right-hand side $b_i(\omega)$, the solution of linear algebraic system of Eqs (20) has the form:

$$\mathrm{B}(\omega) = \mathrm{A}^{-1}(\omega) \cdot \mathrm{C}(\omega), \tag{22}$$

where A^{-1} is the inverse matrix of $\mathrm{A}(\omega)$.

So, all boundary functions are defined, therefore, the Fourier transform (15) for solving the boundary value problem is constructed. Using the inverse Fourier transform (14), we obtain the original $\theta(x,t)$ on the segment [0, *L*].

Let us give as an example the solution of classical BVPs for the heat equation

2.4. Dirichlet problems. In this problem the transformants $\bar{\theta}_j(\omega)\ (j=1,2)$ of temperature at the ends of segment are known. Then boundary conditions (20) take the form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{\theta}_2(\omega) \end{bmatrix}. \tag{23}$$

Substituting these coefficients into system (20), we obtain the solution for determination of unknown heat flow at the ends of the segment:

$$\bar{q}_1(\omega) = \frac{1}{\kappa \bar{U}(L,\omega)} \left\{ \bar{Q}_2(L,\omega) - \kappa \bar{U},_x(L,\omega)\bar{\theta}_1(\omega) - 0{,}5\bar{\theta}_2(\omega) \right\},$$

$$\bar{q}_2(\omega) = -\frac{1}{\kappa \bar{U}(L,\omega)} \left\{ \bar{Q}_1(0,\omega) - 0{,}5\bar{\theta}_1(\omega) - \kappa \bar{U},_x(L,\omega)\bar{\theta}_2(\omega) \right\}.$$

2.5. Neumann problems. The heat flows $\kappa \bar{q}_j(\omega)\ (j=1,2)$ at the ends are known. The boundary conditions (20) for this problem take the form:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix} = \begin{bmatrix} \bar{q}_1(\omega) \\ \bar{q}_2(\omega) \end{bmatrix}.$$

Then

$$\bar{\theta}_j(\omega) = \frac{\Delta_j(\omega)}{\Delta}, \quad j = 1,2,$$

where

$$\Delta = \begin{vmatrix} 0,5 & \kappa\bar{U},_x(L,\omega) \\ \kappa\bar{U},_x(L,\omega) & 0,5 \end{vmatrix} =$$
$$= 0,25 - 0,25\cos^2 k|L| = 0,25(1-\cos^2 k|L|) = 0,25\sin^2 k|L|,$$

$$\Delta_1 = \begin{vmatrix} \bar{Q}_1(0,\omega) + \kappa\bar{U}(L,\omega)\bar{q}_2(\omega) & \kappa\bar{U},_x(L,\omega) \\ \bar{Q}_2(L,\omega) - \kappa\bar{U}(L,\omega)\bar{q}_1(\omega) & 0,5 \end{vmatrix} =$$
$$= 0,5\left\{\bar{Q}_1(0,\omega) + \kappa\bar{U}(L,\omega)\bar{q}_2(\omega)\right\} + \kappa\bar{U},_x(L,\omega)\left\{\bar{Q}_2(0,\omega) - \kappa\bar{U}(L,\omega)\bar{q}_1(\omega\right\}$$

$$\Delta_2 = \begin{vmatrix} 0,5 & \bar{Q}_1(0,\omega) + \kappa\bar{U}(L,\omega)\bar{q}_2(\omega) \\ \kappa\bar{U},_x(L,\omega) & \bar{Q}_2(L,\omega) - \kappa\bar{U}(L,\omega)\bar{q}_1(\omega) \end{vmatrix} =$$
$$= 0,5\left\{\bar{Q}_2(L,\omega) - \kappa\bar{U}(L,\omega)\bar{q}_1(\omega)\right\} - \kappa\bar{U},_x(L,\omega)\left\{\bar{Q}_1(0,\omega) + \kappa\bar{U}(L,\omega)\bar{q}_2(\omega)\right\}.$$

Then $\bar{\theta}_1(\omega) = \dfrac{\Delta_1(\omega)}{\Delta}$, $\bar{\theta}_2(\omega) = \dfrac{\Delta_2(\omega)}{\Delta}$.

2.6. Dirichlet-Neumann problem. The temperature at the left end (x =0) $\bar{\theta}_1(\omega)$ and the heat flow $\bar{q}_2(\omega)$ at the right end ($x = L$) of the form (8) are known .

In this case, the boundary conditions (20) have the form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{b}_3(\omega) \\ \bar{b}_4(\omega) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_2(\omega) \end{bmatrix}.$$

From system (20) we find the unknown boundary values $\bar{\theta}_2(\omega)$ and $\bar{q}_1(\omega)$:

$$\bar{\theta}_2(\omega) = \frac{1}{\kappa\bar{U},_x(L,\omega)}\left\{\bar{Q}_1(0,\omega) - 0,5\bar{\theta}_1(\omega) + \kappa\bar{U}(L,\omega)\bar{q}_2(\omega)\right\},$$

$$\bar{q}_1(\omega) = \frac{1}{\kappa\bar{U}(L,\omega)}\left\{\bar{Q}_2(L,\omega) - \kappa\bar{U},_x(L,\omega)\bar{\theta}_1(\omega) - 0,5\bar{\theta}_2(\omega)\right\}.$$

Here we especially note that the resolving system of equations (20) makes it possible to solve any boundary value problems for the heat equation with local conditions and nonlocal linearly related conditions at the ends of the segment.

3. Algebraic equations for determining unknown boundary functions on a heat star graph

Let's return to the consideration of Dirichlet problem for heat equation

on a star graph (Fig. 1).

On each segment L_j of the graph we have the system of linear algebraic equations for

determining four boundary functions:

$$\begin{pmatrix} 1 & 0 & -\cos(kL_j) & k^{-1}\sin(kL_j) \\ -\cos(kL_j) & -k^{-1}\sin(kL_j) & 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1^j(\omega) \\ \bar{q}_1^j(\omega) \\ \bar{\theta}_2^j(\omega) \\ \bar{q}_2^j(\omega) \end{pmatrix} = \begin{pmatrix} \bar{F}_1^j(\omega) \\ \bar{F}_2^j(\omega) \end{pmatrix}, \qquad (24)$$

where j is the number of the corresponding graph segment, and $\bar{F}_1^j(\omega) = 2\bar{Q}_1^j(0,\omega)$, $\bar{F}_2^j(\omega) = 2\bar{Q}_2^j(L,\omega)$. The graph has N segments with one boundary condition at the end of every segment. Consequently, we have N boundary conditions at the ends of this graph. The next N rows of matrix A contain the conditions of continuity (3) and Kirchhoff (5) for N segments whose ends lie at the vertex of the graph A_0. So we write full system in the next matrix form.

Theorem 2 . *Resolving system of equations of Dirichlet boundary value problem (4) – (5) on a heat star graph with N different segments has the form :*

$$\mathbf{A}(\omega)\times\mathbf{B}(\omega) = \mathbf{C}(\omega), \qquad (25)$$

wher

$$
A = \begin{bmatrix}
1 & 0 & -\cos(k_1L_1) & \frac{\sin(k_1L_1)}{k_1} & .. & 0 & 0 & 0 & 0 \\
-\cos(k_1L_1) & -\frac{\sin(k_1L_1)}{k_1} & 1 & 0 & .. & 0 & 0 & 0 & 0 \\
... & ... & ... & ... & .. & ... & ... & ... & ... \\
... & ... & ... & ... & .. & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & .. & 1 & 0 & -\cos(k_NL_N) & \frac{\sin(k_NL_N)}{k_N} \\
0 & 0 & 0 & 0 & .. & -\cos(k_NL_N) & -\frac{\sin(k_NL_N)}{k_N} & 1 & 0 \\
1 & 0 & 0 & 0 & .. & -1 & 0 & 0 & 0 \\
1 & ... & ... & ... & .. & ... & ... & ... & ... \\
1 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\
1 & ... & ... & ... & .. & ... & ... & ... & ... \\
1 & ... & ... & ... & .. & ... & ... & ... & ... \\
0 & \kappa_1 & 0 & 0 & .. & 0 & 0 & 0 & \kappa_N
\end{bmatrix}
$$

$$\mathbf{B}(\omega) = \left(\bar{\theta}_1^1, \bar{q}_1^1, \bar{\theta}_2^1, \bar{q}_2^1, ..., \bar{\theta}_1^N, \bar{q}_1^N, \bar{\theta}_2^N, \bar{q}_2^N\right),$$

$$\mathbf{C}(\omega) = (\bar{Q}_1^1(0,\omega), \bar{Q}_2^1(L,\omega), ..., \bar{Q}_1^N(0,\omega), \bar{Q}_2^N(L,\omega)), 0, 0, ..., 0, \bar{G}(\omega)).$$

Here the matrices have the following dimensions $\mathbf{A}(\omega) = \{a_{mn}(\omega)\}_{4N\times 4N}$, $\mathbf{B}(\omega) = \{b_{mn}(\omega)\}_{4N\times 1}$.

The first $2N$ rows of matrix A contain the resulting system (19) for each edge of this graph. In matrix A in line ($2N+j$) in column 1 stays 1, in column

($1+4j$) stays the number -1, j =1,…, $2N$-1. In last row in column $2+4j$, j =0,…, $2N$, the value κ_j stays.

The solution of the system (25) has the form:

$$\mathbf{B}(\omega) = \mathbf{A}^{-1}(\omega) \times \mathbf{C}(\omega) \tag{26}$$

After determining the unknown nodal and boundary functions on every edge $\left(\bar{\theta}_1^j, \bar{q}_1^j, \bar{\theta}_2^j, \bar{q}_2^j\right)$, using formulas (6) for $\theta_j(x,\omega)$ on j-th edge (j=1,…,N), we determine the trasformants of temperature of every edges of the graph by use (15), (16):

$$\bar{\theta}_j(x,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}_j(x,\omega) + \theta_0^j(x) H(L-x) H(x) \underset{x}{*} \bar{U}_j(x,\omega) +$$

$$+\kappa\bar{q}_2^j(\omega)H(x)\bar{U}_j(L_j-x,\omega)-\kappa\bar{q}_1^j(\omega)H(L_j-x)\bar{U}_j(x,\omega)- \tag{27}$$

$$-\kappa_j\bar{\theta}_2^j(\omega)H(x)\bar{U}_{j,x}(L_j-x,\omega)-\kappa_j\bar{\theta}_1^j(\omega)H(L_j-x)\bar{U}_{j,x}(x,\omega);$$

where $\bar{U}_j(x,\omega)=-\dfrac{\sin\left(k_j|x|\right)}{2k_j\kappa_j}$, $\bar{U}_{j,x}(x,\omega)=-\dfrac{\operatorname{sgn}x}{2\kappa_j}\cos(\kappa_j|x|)$, $k_j=(1+i)\sqrt{\dfrac{\omega}{2\kappa_j}}$,

$0\le x\le L_j,\quad j=1,2,...,N.$

The original of the solution of BVP are obtained by use (14):

$$\theta_j(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\theta}_j(x,\omega)e^{-i\omega t}d\omega,\quad j=1,2,...,N \tag{28}$$

The boundary value problem on the thermal graph has been solved.

Conclusion

Using the method of generalized functions, boundary value problems of thermal conductivity on a thermal star graph have been solved, which can be used to study various network-like structures under conditions of thermal heating (cooling). A unified technique has been developed for solving various boundary value problems typical for practical applications.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph edge. The obtained regular integral representations of generalized solutions make it possible to determine the temperature and heat flows on each element of the graph, at any point of it, for stationary oscillations with a constant frequency and in the case of periodic oscillations.

For nonstationary processes, performing the inverse Fourier transform in time, we obtain the original solution in the original space-time. The construction of the original depends on the boundary conditions and the type of functions that determine them and should be considered separately for a specific boundary value problem.

The generalized function method presented here makes it possible to solve a wide class of boundary value problems with local and connected boundary conditions at the ends of the edges of the graph and various transmission conditions at its node and can be extended to network structures of very different types. It distinguishes this method from all others that are used to solve similar problems.

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